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Functional integral approach to multipoint correlators in two-dimensional critical systems

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Abstract. We extend a previously developed technique for computing spin–spin critical correlators in the two-dimensional (2D) Ising model, to the case of multiple correlations. This enables us to derive Kadanoff–Ceva's formula in a simple and elegant way. We also exploit a doubling procedure in order to evaluate the critical exponent of the polarization operator in the Baxter model. Thus we provide a rigorous proof of the relation between different exponents, in the path-integral framework.

Since Schultz *et al* [1] showed that Onsager's solution of the two-dimensional (2D) Ising model could be simply explained in terms of a single Majorana fermion, there has been an increasing interest in the study of 2D statistical mechanics models by means of field-theoretical methods. In the same vein, Luther and Peschel [2] proved that the scaling regime of the eight-vertex (Baxter [3]) model can be described in the continuum limit in terms of a Thirring [4] Lagrangian. In this way, the 2D Ising and Baxter models became fruitful testing grounds for new ideas and computational methods. In a previous work it has been shown how to evaluate 2-point correlators in 2D systems [5], through a path-integral approach to bosonization [6]. In particular, the critical behaviour of the Ising (on-line) spin–spin correlation function was obtained, by using a slightly modified version of the identity derived by Zuber and Itzykson [7]:

$$F_2{}^2(x_1, x_2) = \langle\sigma(x_1)\sigma(x_2)\rangle^2 = \left\langle \exp\pi \int_{x_1}^{x_2} \mathrm{d}z \;\; J_0(z) \right\rangle \tag{1}$$

where J_μ is the Dirac fermion current which is obtained out of the original Majorana fields after squaring the correlator. $\langle\rangle$ means vacuum expectation value (VEV) in a model of free massless fermion fields.

The purpose of this paper is twofold. On the one hand, we extend the above-mentioned method to compute the $2n$-point correlator. Thus, we provide an alternative derivation of Kadanoff–Ceva's formula [8] that could be useful when considering certain non-trivial extensions of the Ising model such as the off-critical [9] and the defected [10] cases. On the other hand, we adapt the doubling technique [11] which led to (1), in order to calculate the correlation function of the polarization operator in the Baxter model [3]. This, in turn allows us to provide a path-integral confirmation of the relations between different

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critical exponents (those corresponding to energy density, crossover and polarization), a result previously established by Drugowich de Felicio and Koberle [12] in the operator framework.

For the sake of clarity we shall begin by briefly summarizing the main points of the spin–spin correlator calculation. In [5] the line integral in (1) was written as

$$\int_{x_1}^{x_2} \mathrm{d}z\, J_0(z) = \int \mathrm{d}^2x\, \bar{\Psi}\, \not{A}\Psi$$

where A_μ is an auxiliary vector field with components

$$A_0(z_0, z_1) = \delta(z_0)\theta(z_1 - x_1)\theta(x_2 - z_1)$$
$$A_1(z_0, z_1) = 0.$$

This simple manipulation enabled us to express the squared spin–spin correlator in terms of fermionic determinants

$$F_2{}^2(x_1, x_2) = \frac{\det(\mathrm{i}\not{\partial} + \pi\not{A})}{\det \mathrm{i}\not{\partial}} \tag{2}$$

where the coordinate dependence in the right-hand side of (2) is, of course, contained in $\not{A}$.

Finally, one performs a change of path-integral fermionic variables which is chosen so as to decouple fermions from the background field A_μ. It is interesting to note that, in this formulation, the desired 2-point function is just the square root of the Fujikawa Jacobian J_{F} [13] associated with the transformation in the fermionic measure:

$$F_2(x_1, x_2) = J_{\mathrm{F}}(x_1, x_2)^{1/2}. \tag{3}$$

As shown in [14], this Jacobian must be computed with a gauge-invariant regularization prescription in order to avoid a linear divergence (this gauge invariance is a consequence of a symmetry in the original lattice system [15]). This procedure then leads to the well known power-law decay of the spin–spin on-line function, with exponent equal to $\frac{1}{4}$.

Let us now show how to extend the above depicted technique to the computation of the $2n$-point spin correlation function at criticality. To this end, we follow [16] where it was shown that, after squaring the correlator, each pair of consecutive spin variables can be identified with an exponential similar to the one appearing in (1). (See also [17] for a very interesting study on the doubling procedure and the operator content of fermion fields in the Ising model.) We can then express the squared $2n$-point correlator as

$$F_{2n}{}^2(x_1, \ldots, x_{2n}) = \left\langle \prod_{i=1}^{2n} \sigma(x_i) \right\rangle^2 = \left\langle \prod_{i=1,\mathrm{odd}} \exp \pi \int_{x_i}^{x_{i+1}} \mathrm{d}z\; J_0(z) \right\rangle \tag{4}$$

where, as before, $\langle\rangle$ on the right-hand side means VEV to be evaluated in a model of massless Dirac fermions. It is apparent that each line integral in (4) can be cast in the form

$$\int_{x_i}^{x_{i+1}} \mathrm{d}z\, J_0(z) = \int \mathrm{d}^2z\, J_\mu(z) A_\mu(z; x_i, x_{i+1})$$

where we have introduced the n classical singular potentials

$$A_0(z; x_i, x_{i+1}) = \delta(z_0)\theta(z_1 - x_i)\theta(x_{i+1} - z_1)$$
$$A_1(z; x_i, x_{i+1}) = 0.$$

In order to rewrite (4) in a more compact way we construct a new vector field C_μ as a simple superposition of A_μ's:

$$C_0(z) = \sum_{i=1,\mathrm{odd}}^{2n-1} A_0(z; x_i, x_{i+1}) \tag{5}$$

$$C_1(z) = 0. \tag{6}$$

Thus, the $2n$-point function can be expressed in terms of fermionic determinants

$$F_{2n}{}^2 = \frac{\det(\mathrm{i}\partial\!\!\!/ + \pi C\!\!\!/)}{\det \mathrm{i}\partial\!\!\!/} \tag{7}$$

exactly as it happens in the $n = 1$ case (see (2)), but with A_μ replaced by C_μ.

The next step is to write C_μ in terms of scalar functions Φ_c and η_c as

$$C_\mu = \epsilon_{\mu\nu}\partial_\nu\Phi_c + \partial_\mu\eta_c. \tag{8}$$

Now we perform a decoupling change of path-integral fermionic variables with chiral and gauge parameters Φ_c and η_c, respectively:

$$\Psi = \mathrm{e}^{-\pi(\gamma_5\Phi_c + \mathrm{i}\eta_c)}\chi \tag{9}$$

$$\bar\Psi = \bar\chi\ \mathrm{e}^{-\pi(\gamma_5\Phi_c - \mathrm{i}\eta_c)}. \tag{10}$$

A detailed computation of the Fujikawa Jacobian J_{F} associated to this change has been given many times in the literature (see, for instance, [6]); here we just write down the final result:

$$J_{\mathrm{F}} = \exp\frac{\pi}{2}\int \mathrm{d}^2x\ \Phi_c\,\Box\,\Phi_c. \tag{11}$$

We then get

$$F_{2n}{}^2(x_1, \ldots, x_{2n}) = J_{\mathrm{F}}(x_1, x_2, \ldots, x_{2n}). \tag{12}$$

Therefore we see that, in our formulation, the squared multipoint correlator can be identified with a fermionic Jacobian, exactly as in the 2-point case. At this stage one has to solve the system of differential equations for Φ_c and η_c, obtained by replacing (8) in (5) and (6). Finally, by inserting the result in (11) and (12), one obtains

$$F_{2n}(x_1, \ldots, x_{2n}) = \left(\frac{\prod_{\mathrm{even}}|x_{ij}|}{\prod_{\mathrm{odd}}|x_{ij}|}\right)^{1/4} \tag{13}$$

where $i > j$ and even (odd) refers to a constraint on $i + j$; $i, j = 1, 2, \ldots, 2n$. We have also set an ultraviolet cut-off, which divides the coordinate differences, equal to 1. This formula exactly coincides with the famous Kadanoff–Ceva's result [8].

Let us now study the Baxter model [3], which can be considered as two Ising systems interacting through their spin variables (this model is related, through a duality transformation, to the Ashkin–Teller model [18]). As shown by Luther and Peschel [2], the scaling limit of this model is described by the Thirring [4] interaction

$$\mathcal{L}_{\mathrm{int}} = -\lambda J_\mu J_\mu \tag{14}$$

where, as before, J_μ is the Dirac fermionic current and the coupling constant λ is proportional to the four-spin coupling of the original lattice model. The Baxter model is known to have two natural order parameters, the magnetization and the polarization $\langle P\rangle = \langle\sigma_i s_i\rangle$, where σ_i and s_i are the spin operators of each Ising system. In the continuous formulation the 2-point correlator for the polarization operator is given by

$$\langle P(x)P(y)\rangle_\lambda = \langle\sigma_x s_x \sigma_y s_y\rangle_\lambda$$

where $\langle\rangle_\lambda$ means VEV with respect to the fermionic model defined by (14). For $\lambda = 0$ the above expression becomes the squared Ising correlator. This suggests the following identification:

$$\langle P(0)P(R)\rangle_\lambda = \left\langle \exp\ \pi\int_0^R \mathrm{d}z\ J_0(z)\right\rangle_\lambda.$$

The right-hand side of the precedent equation can be computed by employing a slightly modified version of the method described above. Indeed, it is easy to show that the introduction of an auxiliary vector field A_μ through a Hubbard–Stratonovich identity, allows one to write

$$\langle P(0)P(R)\rangle_\lambda = \frac{Z}{Z'} \tag{15}$$

with

$$Z = \int \mathcal{D}A_\mu \exp\left[-\int d^2x \frac{A^2}{2}\right] \det(i\partial\!\!\!/ + (2\lambda)^{1/2} B\!\!\!/) \tag{16}$$

and

$$Z' = \int \mathcal{D}A_\mu \exp\left[-\int d^2x \frac{A^2}{2}\right] \det(i\partial\!\!\!/ + (2\lambda)^{1/2} A\!\!\!/) \tag{17}$$

where

$$B_\mu = \epsilon_{\mu\nu}\partial_\nu \Phi_B + \partial_\mu \eta_B$$
$$A_\mu = \epsilon_{\mu\nu}\partial_\nu \Phi + \partial_\mu \eta$$
$$\Phi_B = \Phi + \frac{\pi}{\sqrt{2\lambda}} \Phi_c$$
$$\eta_B = \eta + \frac{\pi}{\sqrt{2\lambda}} \eta_c.$$

Let us stress that, in contrast to the previous calculation of the Ising correlator, in the present case one has to consider quantum fields Φ and η whose dynamics plays a crucial role in the following computation. Concerning the classical functions Φ_c and η_c, they can be determined exactly as in the Ising case, i.e. using formulae (5), (6) and (8) for $n = 1$.

We shall now turn to treat the fermionic determinants appearing in (16) and (17) by means of decoupling changes of fermionic variables, similar to the one defined by equations (9) and (10), but with parameters Φ_B and η_B in the form:

$$\Psi = e^{-\sqrt{2\lambda}(\gamma_5 \Phi_B + i\eta_B)} \chi$$
$$\bar{\Psi} = \bar{\chi}\, e^{-\sqrt{2\lambda}(\gamma_5 \Phi_B - i\eta_B)}.$$

The corresponding Jacobian is given by

$$J_F = \exp \frac{\lambda}{\pi} \int d^2x \left(\Phi + \frac{\pi}{\sqrt{2\lambda}} \Phi_c\right) \Box \left(\Phi + \frac{\pi}{\sqrt{2\lambda}} \Phi_c\right).$$

Of course, this result must be used in (16), whereas the same expression, but with $\Phi_c = 0$ is to be employed in (17). In so doing one readily discovers that, due to the fact that J_F does not depend on the field η, this field becomes decoupled from Φ in both Z and Z'. As the corresponding functional integrals over η coincide, they cancelled out when performing the quotient in equation (15) and one then gets

$$\langle P(0)P(R)\rangle_\lambda = \langle P(0)P(R)\rangle_0 \left\langle \exp\left[\sqrt{2\lambda} \int d^2x\, \Phi \partial_\mu \partial_\mu \Phi_c\right]\right\rangle$$

where the first factor on the right-hand side corresponds to the doubled Ising correlator, whereas the second one is a VEV to be evaluated for a model of free scalars Φ with Lagrangian density given by

$$\mathcal{L} = \left(\frac{1}{2} + \frac{\lambda}{\pi}\right) \partial_\mu \Phi \partial_\mu \Phi.$$

As it is well known this computation can be done by a standard shift in the bosonic variable Φ. The final result is

$$\langle P(0)P(R)\rangle_\lambda = \left(\frac{a}{R}\right)^{2\Delta_P} \tag{18}$$

where a is an ultraviolet cut-off and Δ_P is the critical exponent associated with the polarization operator, for which we obtain

$$\Delta_P = \frac{1}{4}\frac{1}{(1+(2\lambda/\pi))}. \tag{19}$$

Recalling the results for the energy density (ϵ) and the crossover (Cr) operators [5, 12], one obtains

$$4\Delta_P = \Delta_\epsilon = (\Delta_{Cr})^{-1} \tag{20}$$

which is the relation predicted by several authors [19, 20] and first derived by Drugowich de Felicio and Koberle [12] in the operator framework.

In summary, we have extended a functional approach [5], previously used to compute 2-point functions in 2D critical systems, to the case in which multipoint correlators are considered. In particular, we provided an alternative derivation of Kadanoff and Ceva's result [8] for the $2n$-spin on-line function. Our contribution can be viewed as a complement to previous works based on operational bosonization, where 4-point functions were explicitly calculated [7, 21]. We feel that our formulation could be more practical when considering, for instance, non-critical correlations [9]. Indeed, in this case one expects to have a temperature-dependent ('massive') determinant, that can be easily handled by following the perturbative strategy of [22]. The study of multipoint correlators in the defected Ising model [10] can be also envisaged in our scheme.

We have also computed the 2-point function describing the critical fluctuations of the Baxter polarization operator. Thus we obtained its corresponding critical index. This completed the path-integral proof of the relationship between energy density, crossover and polarization exponents, which had been initiated in [5].

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